

Ch 4.1

MA121

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Next week's MT2 on Wednesday, June 6th will cover Section 2.3 - 3.5.

Continuing from yesterday's thought...

8p106 What if Anastasia's stock has a 4% decline for 3 years followed by a 4% growth over 3 years?

$P = \$80$, $r = -4\% = -0.04$ over $t = 3$ years.

$$F_1(3) = \$80(1 - 0.04)^3 \approx \$70.78$$

over the next 3 years, $P = \$70.78$, $r = 4\% = 0.04$, $t = 3$

$$F_2(3) = \$70.78(1 + 0.04)^3 \approx \$79.62$$

In Summary,

P = Principle, r_{eff} = effective rate of growth over t = number of years.

$$F_1(t) = P(1 + r_{\text{eff}})^t$$

over the next phase of the same t number of years with $-r_{\text{eff}}$ rate of change.

$$F_2(t) = F_1(t)(1 - r_{\text{eff}})^t$$

$$\Rightarrow F_2(t) = P(1 + r_{\text{eff}})^t(1 - r_{\text{eff}})^t$$

$$\Rightarrow F_2(t) = P[(1 + r_{\text{eff}})(1 - r_{\text{eff}})]^t$$

$$\Rightarrow F_2(t) = P(1 - r_{\text{eff}}^2)^t$$

4.2 Compound Interest

In the previous section, we discussed cases where the interest is added only once a year. This means the interest is compounded annually.

$$F(t) = P(1 + r_{\text{eff}})^t \quad \text{where } r = r_{\text{eff}}$$

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If the compound occurs more than once a year,

↳ r is no longer the effective rate. It becomes the ~~nominal~~ rate.

↳ The formula for the future value becomes

$$F(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where: $F(t)$ = amount after t years

P = Principle

r = nominal interest rate

t = # of years

n = # of compounds per year.

EX 4p 110

If Mario invests \$25,000 in a premium rate savings account that pays a nominal rate of 6.25% compounded monthly for 7.5 years, how much will Mario withdraw at maturity? What is the total amount of interest earned?

Given $P = \$25,000$, $r = 6.25\% = 0.0625$, $t = 7.5$, $n = 12$

$$F(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$F(7.5) = 25,000\left(1 + \frac{0.0625}{12}\right)^{(12)(7.5)}$$

$$\Rightarrow F(7.5) = 25,000\left(1 + \frac{0.0625}{12}\right)^{90}$$

$$\Rightarrow F(7.5) \approx 25,000(1 + 0.005208333...)^{90}$$

$$\Rightarrow F(7.5) \approx 25,000(1.5960527)$$

$$\Rightarrow F(7.5) \approx \$39,901.32$$

Amount of Interest earned =

$$\$39,901.32 - \$25,000 = \$14,901.32$$

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Sp110

If Luigi's Stock of gold declines at a nominal rate of 4%/year compounded SEMI-ANNUALLY for 12 months to a value of \$106.50, what was the original value of the stock?

Given $r = -4\% = -0.04$, $t = 1$ (years), $n = 2$ (semi annual)

$$F(1) = \$106.50$$

We want to find P .

$$F(t) = P \left(1 + \frac{r}{n} \right)^{nt}$$

$$\Rightarrow 106.50 = P \left(1 + \frac{-0.04}{2} \right)^{2(1)}$$

$$\Rightarrow 106.50 = P(1 - 0.02)^2$$

$$\Rightarrow 106.50 = P(0.98)^2$$

$$\Rightarrow 106.50 = P(0.9604)$$

$$\Rightarrow P = \frac{106.50}{0.9604}$$

$$\approx \$110.89$$

Effective Interest rate

If we keep our investment for a complete year without touching the account, then the amount earned at the end of the completed year tells us the effective interest rate.

$$\text{Recall } r_{\text{eff}} = \frac{V_1 - V_0}{V_0} = \frac{V(t_0+1) - V(t_0)}{V(t_0)}$$

Where $V_1 = V(t_0+1)$ is the amount in the account after one complete year

$$\text{Comparing } F(t) = P \left(1 + \frac{r}{n} \right)^{tn} \text{ and } F(t) = P(1 + r_{\text{eff}})^t,$$

$$\text{we have } 1 + r_{\text{eff}} = \left(1 + \frac{r}{n} \right)^n$$

$$\Rightarrow \boxed{r_{\text{eff}} = \left(1 + \frac{r}{n} \right)^n - 1}$$

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Ex 1 P110 Find the effective rate of interest when the nominal rate $r = 5.25\%$ is compounded semi-annually, quarterly, and weekly.

$$\text{given } r = 5.25\% = 0.0525$$

i) When $n = 2$,

$$r_{\text{eff}} = \left(1 + \frac{r}{n}\right)^n - 1$$

$$= \left(1 + \frac{0.0525}{2}\right)^2 - 1$$

$$= (1 + 0.02625)^2 - 1$$

$$= (1.02625)^2 - 1$$

$$r_{\text{eff}} = 0.05318, \text{ or } 5.318\%$$

ii) When $n = 4$,

$$r_{\text{eff}} = \left(1 + \frac{r}{n}\right)^n - 1$$

$$= \left(1 + \frac{0.0525}{4}\right)^4 - 1$$

$$= (1 + 0.0131)^4 - 1$$

$$= (1.0131)^4 - 1$$

$$r_{\text{eff}} = 0.0535 \text{ or } 5.35\%$$

iii) When $n = 52$,

$$r_{\text{eff}} = \left(1 + \frac{r}{n}\right)^n - 1$$

$$= \left(1 + \frac{0.0525}{52}\right)^{52} - 1$$

$$= (1 + 0.00100)^{52} - 1$$

$$= (1.001)^{52} - 1$$

$$= 0.0539 \text{ or } 5.39\%$$

Continuous Compounding

Let's analyse what happens when the number of compounds per year increases to infinity. In other words, what happens when the compound becomes Continuous.

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We have $F(t) = P(1 + \frac{r}{n})^{nt}$ when $n = \infty$

$$F(t) = P \lim_{n \rightarrow \infty} (1 + \frac{r}{n})^{nt}$$

$$\text{Let } \frac{r}{n} = \frac{1}{m}$$

$\Rightarrow n = mr$ so that as $n \rightarrow \infty$, we have $m = \infty$.
the formula becomes

$$F(t) = P \lim_{m \rightarrow \infty} (1 + \frac{1}{m})^{mrt}$$

Let's see what happens when $\lim_{m \rightarrow \infty} (1 + \frac{1}{m})^m$
(refer to table on p108)

As $m \rightarrow \infty$, we have $(1 + \frac{1}{m})^m \rightarrow 2.7182818 \dots$

This limiting value is a very famous constant, e .

Definition of e .

$$e = \lim_{m \rightarrow \infty} (1 + \frac{1}{m})^m \quad e \approx 2.718281828459 \dots$$

Now we can rewrite our formula for continuous compounding,

$$F(t) = P \lim_{m \rightarrow \infty} (1 + \frac{r}{n})^{nt} = P \lim_{m \rightarrow \infty} (1 + \frac{1}{m})^{mrt}$$

$$\Rightarrow F(t) = P e^{rt}$$

The effective interest rate is

$$r_{\text{eff}} = \lim_{m \rightarrow \infty} (1 + \frac{r}{n})^n - 1 = \lim_{m \rightarrow \infty} (1 + \frac{1}{m})^{mr} - 1$$

$$\Rightarrow r_{\text{eff}} = e^r - 1 \quad \leftarrow \text{for continuous compounding.}$$

ex. 8p110 You have the opportunity to invest your income tax return at a nominal rate of 12% with quarterly

12.5% with annual compounding, or 11% with continuous compounding. Which option yields the best effective rate? Using the best nominal rate, what will be the future value if you invest your \$850 tax return for 4 years?

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i) $r = 12\%$ when $n = 4$

ii) $r = 12.5\%$ when $n = 1$

iii) $r = 11\%$ when continuous compounding.

i) $r_{\text{eff}} = \left(1 + \frac{r}{n}\right)^n - 1$ when $n = 4$

$$\Rightarrow r_{\text{eff}} = \left(1 + \frac{0.12}{4}\right)^4 - 1$$

$$\approx 12.550881\ldots\%$$

ii) $r_{\text{eff}} = r = 12.5\%$ since $n = 1$

iii) $r_{\text{eff}} = e^r - 1$

$$= e^{0.11} - 1$$

$$\approx 11.627807\ldots\%$$

in conclusion, option i yields the highest effective rate, which yields the highest return.

PART 2) $P = 850$ $t = 4$

$$F(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$\Rightarrow F(4) = 850\left(1 + \frac{0.12}{4}\right)^{(4)(4)}$$

$$= \$1364.00$$

OR $F(t) = P(1 + r_{\text{eff}})^t$

$$F(4) = 850(1 + 0.1255)^4$$

$$= \$1364.00$$

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ex 10 p 110/111

Ash and Brock take out a \$15000 bank loan. After 4 years they pay back \$17120 (the principal plus \$2120 in interest). Determine the nominal rate of r of their loan if it was compounded semi-annually.

Given $P = \$15,000$, $t = 4$ years, $F(4) = \$17,120$
(interest = \$2120).

$$F(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \leftarrow \text{want to find } r$$

$$17120 = 15000\left(1 + \frac{r}{2}\right)^{(2)(4)}$$

$$\Rightarrow \frac{17120}{15000} = \left(1 + \frac{r}{2}\right)^8$$

$$\Rightarrow \sqrt[8]{\frac{17120}{15000}} = 1 + \frac{r}{2}$$

$$\Rightarrow \frac{r}{2} = \sqrt[8]{\frac{17120}{15000}} - 1$$

$$\Rightarrow r = 2\left(\sqrt[8]{\frac{17120}{15000}} - 1\right)$$

$$\Rightarrow r = 0.033238, \dots$$

$$\Rightarrow r \approx 3.32\%$$